Notes for Final Exam

Math 250:B1, Summer 2003

Rotation Matrices:
$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Matrix Algebra:
$(AB)^T = B^T A^T$
$A(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) = c_1 A\mathbf{u}_1 + c_2 A\mathbf{u}_2$
$(AB)^{-1} = B^{-1}A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$

Algorithm to find A^{-1}: $[A\ I] \xrightarrow{\text{rref}} [I\ A^{-1}]$.

If an augmented matrix $[A\ \mathbf{b}]$ contains a row where the only nonzero entry is in the last column, then $A\mathbf{x} = \mathbf{b}$ is inconsistent.

The rank of A is the number of nonzero rows in the rref of A. The nullity of $A_{m\times n}$ is $n - \text{rank}\, A$.

$A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

rank of $A_{m\times n}$	number of sols to $A\mathbf{x} = \mathbf{b}$	columns of A	rref of A
m	at least one for every $\mathbf{b} \in \mathbb{R}^m$	spanning set for $\mathbb{R}^m$	every row contains a pivot position
n	at most one for every $\mathbf{b} \in \mathbb{R}^m$	linearly independent	every column contains a pivot position

Performing an elementary row operation on A is the same as multiplying A by the corresponding elementary matrix E.

If R is the rref of $A_{m\times n}$, then there exists an invertible matrix $P_{m\times m}$ such that $PA = R$.

Linear Correspondence Property: Any linear relationship of the columns of A also applies to the columns of rref A, and vice versa.

$A_{n\times n}$ is invertible if and only if rref of A is $I_{n\times n}$.

If A can be put into ref form without using row swaps, then A can be written as LU, where L is a unit lower triangular matrix, and U is an upper triangular matrix. If $U = E_k \cdots E_2 E_1 A$, then $L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.

$A_{n\times n}$ is invertible if and only if $\det A \neq 0$.

$\det(AB) = (\det A)(\det B)$

subspace from $A_{n\times n}$	dimension	basis
Col A	rank A	pivot columns of A
Row A	rank A	nonzero rows of rref of A
Null A	nullity A	vectors in parametric solution of $A\mathbf{x} = \mathbf{0}$

λ is an eigenvalue of $A_{n\times n}$ if there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

The characteristic polynomial of A is $\det(A - tI)$.

The dimension of the eigenspace corresponding to an eigenvalue λ is at most the multiplicity of λ.

$A_{n\times n}$ is diagonalizable if there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of A. Then $A = PDP^{-1}$, where P has the eigenvectors as columns and D is a diagonal matrix with the corresponding eigenvalues along the diagonal.

Quadratic Formula: If $at^2 + bt + c = 0$, then $t = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. If $b^2 - 4ac < 0$, then the roots of the polynomial are complex.

A Markov chain with a finite number of states is regular if it is possible when starting from some state x to eventually move to any other state. A sufficient condition for the Markov chain to be regular is that the transition matrix has no zero entries. If A is the transition matrix for a regular Markov chain, then 1 is an eigenvalue of A and there is a unique probability vector $\mathbf{p}$ that is also an eigenvector corresponding to 1. The limit of $A^m\mathbf{v}$ as $m \to \infty$ and for any probability vector $\mathbf{v}$ is $\mathbf{p}$.

Let B be a matrix whose columns form a basis $\mathcal{B}$ for $\mathbb{R}^n$. Then for every vector $\mathbf{v} \in \mathbb{R}^n$, $B[\mathbf{v}]_{\mathcal{B}} = \mathbf{v}$, and since B is invertible, $[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$.

$\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$

Pythagorean Theorem: $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal.

Orthogonal Projection of $\mathbf{v}$ onto span$\{\mathbf{u}\}$: $\dfrac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2}\mathbf{u}$

Gram-Schmidt Orthogonalization: Start with a linearly indepedent set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq \mathbb{R}^n$. Then $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set where span $S' =$ span S.

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 \\ \mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 \\ &\vdots \\ \mathbf{v}_k &= \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \ldots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{||\mathbf{v}_{k-1}||^2}\mathbf{v}_{k-1} \end{aligned}$$

The orthogonal complement $S^\perp$ is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to each vector in a nonempty set $S \subseteq \mathbb{R}^n$.

Closest Vector Property: The closest vector in a subspace $W \subseteq \mathbb{R}^n$ to $\mathbf{v}$ is the orthogonal projection of $\mathbf{v}$ onto W.

Method of Least Squares: To find the line $y = a_0 + a_1 x$ that best fits the data, let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Let $C = [\mathbf{v}_1 \ \mathbf{v}_2]$. Then

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = (C^T C)^{-1} C^T \mathbf{y}$$

The orthogonal projection matrix P_W can be found by computing $C(C^T C)^{-1} C^T$, where the columns of C are a basis for the subspace W.

$A_{n \times n}$ is symmetric if and only if $A = PDP^{-1}$, where P is orthogonal and D is diagonal.

Spectral Decomposition of a Symmetric Matrix: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of a symmetric matrix A. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. Then

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$