Notes for Final Exam

Math 250:B1, Summer 2003

Rotation Matrices:
\[ A_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

Matrix Algebra:
\[ (AB)^T = B^T A^T \]
\[ A(c_1 u_1 + c_2 u_2) = c_1 A u_1 + c_2 A u_2 \]
\[ (AB)^{-1} = B^{-1} A^{-1} \]
\[ (A^T)^{-1} = (A^{-1})^T \]

Algorithm to find \( A^{-1} \): \([ A \ I \] \xrightarrow{\text{rref}} [ I \ A^{-1} ]\).

If an augmented matrix \([ A \ b ]\) contains a row where the only nonzero entry is in the last column, then \( Ax = b \) is inconsistent.

The rank of \( A \) is the number of nonzero rows in the rref of \( A \). The nullity of \( A_{m \times n} \) is \( n - \text{rank } A \).

\( Ax = b \) is consistent if and only if \( b \in \text{span}\{a_1, a_2, \ldots, a_n\} \).

<table>
<thead>
<tr>
<th>\text{rank of } A_{m \times n}</th>
<th>\text{number of sols to } Ax = b</th>
<th>\text{columns of } A</th>
<th>\text{rref of } A</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>at least one for every ( b \in \mathbb{R}^m )</td>
<td>spanning set for ( \mathbb{R}^m )</td>
<td>every row contains a pivot position</td>
</tr>
<tr>
<td>( n )</td>
<td>at most one for every ( b \in \mathbb{R}^m )</td>
<td>linearly independent</td>
<td>every column contains a pivot position</td>
</tr>
</tbody>
</table>

Performing an elementary row operation on \( A \) is the same as multiplying \( A \) by the corresponding elementary matrix \( E \).

If \( R \) is the rref of \( A_{m \times n} \), then there exists an invertible matrix \( P_{m \times m} \) such that \( PA = R \).

Linear Correspondence Property: Any linear relationship of the columns of \( A \) also applies to the columns of rref \( A \), and vice versa.

\( A_{n \times n} \) is invertible if and only if rref of \( A \) is \( I_{n \times n} \).

If \( A \) can be put into ref form without using row swaps, then \( A \) can be written as \( LU \), where \( L \) is a unit lower triangular matrix, and \( U \) is an upper triangular matrix. If \( U = E_k \cdots E_2 E_1 A \), then \( L = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \).

\( A_{n \times n} \) is invertible if and only if \( \det A \neq 0 \).

\[ \det(AB) = (\det A)(\det B) \]

Subspace from \( A_{n \times n} \)

<table>
<thead>
<tr>
<th>Col A</th>
<th>rank A</th>
<th>pivot columns of A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row A</td>
<td>rank A</td>
<td>nonzero rows of rref of A</td>
</tr>
<tr>
<td>Null A</td>
<td>nullity A</td>
<td>vectors in parametric solution of ( Ax = 0 )</td>
</tr>
</tbody>
</table>

\( \lambda \) is an eigenvalue of \( A_{n \times n} \) if there is a nonzero vector \( v \) such that \( Av = \lambda v \).

The characteristic polynomial of \( A \) is \( \det(A - \lambda I) \).

The dimension of the eigenspace corresponding to an eigenvalue \( \lambda \) is at most the multiplicity of \( \lambda \).

\( A_{n \times n} \) is diagonalizable if there exists a basis for \( \mathbb{R}^n \) consisting of eigenvectors of \( A \). Then \( A = PDP^{-1} \), where \( P \) has the eigenvectors as columns and \( D \) is a diagonal matrix with the corresponding eigenvalues along the diagonal.

Quadratic Formula: If \( at^2 + bt + c = 0 \), then \( t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \). If \( b^2 - 4ac < 0 \), then the roots of the polynomial are complex.

A Markov chain with a finite number of states is regular if it is possible when starting from some state \( x \) to eventually move to any other state. A sufficient condition for the Markov chain to be regular is that the transition matrix has no zero entries. If \( A \) is the transition matrix for a regular Markov chain, then \( 1 \) is an eigenvalue of \( A \) and there is a unique probability vector \( p \) that is also an eigenvector corresponding to \( 1 \). The limit of \( A^n v \) as \( n \to \infty \) and for any probability vector \( v \) is \( p \).

Let \( B \) be a matrix whose columns form a basis \( B \) for \( \mathbb{R}^n \). Then for every vector \( v \in \mathbb{R}^n \), \( B[v]_B = v \), and since \( B \) is invertible, \( [v]_B = B^{-1} v \).
\[ \mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2 \]

Pythagorean Theorem: \( ||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \) if and only if \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^n \) are orthogonal.

Orthogonal Projection of \( \mathbf{v} \) onto \( \text{span}\{\mathbf{u}\} \): \( \frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2} \mathbf{u} \)

Gram-Schmidt Orthogonalization: Start with a linearly independent set \( S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq \mathbb{R}^n \). Then \( S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \) is an orthogonal set where \( \text{span} \ S' = \text{span} \ S \).

\[
\begin{align*}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2} \mathbf{v}_1 \\
& \quad \vdots \\
\mathbf{v}_k &= \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2} \mathbf{v}_1 - \cdots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{||\mathbf{v}_{k-1}||^2} \mathbf{v}_{k-1}
\end{align*}
\]

The orthogonal complement \( S^\perp \) is the set of all vectors in \( \mathbb{R}^n \) that are orthogonal to each vector in a nonempty set \( S \subseteq \mathbb{R}^n \).

Closest Vector Property: The closest vector in a subspace \( W \subseteq \mathbb{R}^n \) to \( \mathbf{v} \) is the orthogonal projection of \( \mathbf{v} \) onto \( W \).

Method of Least Squares: To find the line \( y = a_0 + a_1 x \) that best fits the data, let

\[
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
= 
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix},
\quad
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
\]

Let \( C = [\mathbf{v}_1 \mathbf{v}_2] \). Then

\[
\begin{bmatrix}
a_0 \\
a_1
\end{bmatrix} = (C^T C)^{-1} C^T y
\]

The orthogonal projection matrix \( P_W \) can be found by computing \( C(C^T C)^{-1} C^T \), where the columns of \( C \) are a basis for the subspace \( W \).

\( A_{n \times n} \) is symmetric if and only if \( A = PDP^{-1} \), where \( P \) is orthogonal and \( D \) is diagonal.

Spectral Decomposition of a Symmetric Matrix: Let \( \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \) be an orthonormal basis for \( \mathbb{R}^n \) consisting of eigenvectors of a symmetric matrix \( A \). Let \( \lambda_1, \ldots, \lambda_n \) be the corresponding eigenvalues. Then

\[
A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.
\]

\[ 2 \]