MATH 103, COLLEGE ALGEBRA — USEFUL FORMULAS

Review and Chapter 1 — Basic Concepts of Algebra, Functions, Graphs

Properties of Exponents

1. $a^m \cdot a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
6. $a^{\frac{1}{n}} = \sqrt[n]{a}$

Special Products

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)(a-b) = a^2 - b^2$
4. $(a+b)^3 = (a+b)(a^2 - ab + b^2)$
5. $(a-b)^3 = (a-b)(a^2 + ab + b^2)$

Variation

1. Direct: $y = kx$
2. Inverse: $y = \dfrac{k}{x}$
3. Joint: $y = kxz$

Linear Functions

1. Slope-Intercept Eq. $y = mx + b$
2. Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
3. Point-Slope Eq. $y - y_1 = m(x - x_1)$
4. Parallel Lines — Equal Slopes
5. Perpendicular Lines $m_1 m_2 = -1$

Domain

1. $\dfrac{1}{x-a}$: $\mathbb{R} \setminus \{a\}$
2. $\sqrt{x-a}$: $[a, +\infty)$
3. $f \pm g, f \cdot g$: all values of x that are in both domains of f and g
4. $\dfrac{f}{g}$: all values of x that are in both domains of f and g and $g(x) \neq 0$

Different Formulas

1. Distance: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. Midpoint: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$
3. Circle: $(x-h)^2 + (y-k)^2 = r^2$
4. Difference Quotient: $\dfrac{f(x+h) - f(x)}{h}$

Tests for Symmetry

1. x-axis : change (x, y) to $(x, -y)$
2. y-axis : change (x, y) to $(-x, y)$
3. Origin : change (x, y) to $(-x, -y)$
4. Even function : $f(-x) = f(x)$
5. Odd function : $f(-x) = -f(x)$

Transformations

1. $f(x) + b$: shift **up** b units
2. $f(x) - b$: shift **down** b units
3. $f(x + b)$: shift **left** b units
4. $f(x - b)$: shift **right** b units
5. $bf(x)$: stretch vertically
6. $\dfrac{f(x)}{b}$: shrink vertically
7. $f(bx)$: shrink horizontally
8. $f\left(\dfrac{x}{b}\right)$: stretch horizontally
9. $-f(x)$: reflection across x-axis
10. $f(-x)$: reflection across y-axis

Chapter 2 — Functions and Equations, Linear and Quadratic Functions

Zeros of Functions

1. x_0 is a zero of f iff $f(x_0) = 0$
2. **Real** zero — x-intercept
3. The only zero of a **linear** function $f(x) = mx + b$ is $x = -\dfrac{b}{m}$, iff $m \neq 0$
4. **Quadratic Formula** Zeros of $f(x) = ax^2 + bx + c,\ a \neq 0$:
   $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
   Discriminant $D = b^2 - 4ac$
   $D > 0$ — two different real zeros
   $D = 0$ — one double zero
   $D < 0$ — two complex zeros

Parabola

for $f(x) = a(x-h)^2 + k$

1. Vertex: (h, k)
2. Axis of symmetry: $x = h$
3. $a > 0$: k— minimum, $[k, +\infty)$— range
4. $a < 0$: k— maximum, $(-\infty, k]$— range
5. $f(x) = ax^2 + bx + c$: $h = -\dfrac{b}{2a}$, $k = f(h)$.

Equations

1. Rational: multiply by LCD.
2. Radical: isolate radical and rise both sides to the corresponding power.
3. $|X| = a$ for $a \geqslant 0$ is equivalent to $X = a$ OR $X = -a$
4. $|X| = a$ for $a < 0$ has no solutions

Complex Numbers

1. $i = \sqrt{-1}$ $\qquad$ $\sqrt{-a} = i\sqrt{a}$, **if** $a > 0$
2. $i^2 = -1,\ i^3 = -i,\ i^4 = 1$
3. Complex Number: $a + bi,\ a, b \in \mathbb{R}$, a — real part, b — imaginary part.
4. Complex Conjugate $\overline{a + bi} = a - bi$
5. $\dfrac{a+bi}{c+di} = \dfrac{(a+bi)(c-di)}{(c+di)(c-di)} = \dfrac{(a+bi)(c-di)}{c^2+d^2}$
6. $6 = \sqrt{(-9)\cdot(-4)} \neq \sqrt{-9}\cdot\sqrt{-4} = -6$

Different Formulas

1. Uniform motion: $d = vt$, v — speed, t — time , d — distance.

Completing the Square

1. $a'x^2 + b'x + c' = 0$
2. $x^2 + bx + c = 0$
3. $x^2 + 2 \cdot x \cdot \dfrac{b}{2} + \left(\dfrac{b}{2}\right)^2 = -c + \left(\dfrac{b}{2}\right)^2$
4. $\left(x + \dfrac{b}{2}\right)^2 = \left(\dfrac{b}{2}\right)^2 - c$
5. $x + \dfrac{b}{2} = \pm\sqrt{\left(\dfrac{b}{2}\right)^2 - c}$

Inequalities

1. $a < b$ is equivalent to $a + c < b + c$
2. $a < b$ is equivalent to $ac < bc$, **if** $c > 0$
3. $a < b$ is equivalent to $ac > bc$, **if** $c < 0$
4. Conjunction — AND — Intersection $\cap$
5. Disjunction — OR — Union $\cup$
6. $|X| < a$ is equivalent to $X < a$ AND $X > -a$
7. $|X| > a$ is equivalent to $X < -a$ OR $X > a$

Chapter 3 — Polynomial and Rational Functions, Inequalities

Polynomials

1. $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$
2. a_nx^n — leading term
3. a_n — leading coefficient
4. $n=\deg f(x)$ — degree
5. Domain — all real numbers
6. Has at most n x-intercepts
7. Has at most $n-1$ turning points
8. If $f(a)$ and $f(b)$ have different signs, then $f(x)$ has a zero between a and b

Theorems about Zeros (Roots)

1. $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$
2. nth degree polynomial has n complex roots (counting multiplicity)
3. $f(x)=a_n(x-c_1)(x-c_2)\ldots(x-c_n)$
4. If $a+bi$ is a root and all coefficients are **real**, then $a-bi$ is also a root
5. If $a+c\sqrt{b}$ is a root, a, c are rational, $\sqrt{b}$ is irrational and all coefficients are **rational**, then $a-c\sqrt{b}$ is also a root
6. If $\frac{p}{q}$ is a root in the lowest terms and all coefficients are **integer**, then p is a factor of a_0 and q is a factor of a_n

Rational Functions

1. $\frac{p(x)}{q(x)}$: $p(x), q(x)$ are polynomials
2. Domain — all real numbers except for zeros of the denominator $q(x)$
3. **Factor** $p(x)$ and $q(x)$ and write $\frac{p(x)}{q(x)}$ in the **lowest terms**
4. **Real** zeros of $q(x)$: vertical asymp.
5. **Real** zeros of $p(x)$: x-intercepts
6. $\deg p<\deg q$: horizontal asymp. $y=0$
7. $\deg p=\deg q$: horizontal asymp.
8. $\deg p=\deg q+1$: oblique asymp.

Inequalities

1. Get zero on the right hand side
2. Write as a single fraction
3. Find zeros of both numerator and denominator
4. Mark **real** zeros on the number line
5. Select test points and determine the sign of the expression in each interval
6. Strict inequality : exclude endpoints
7. Non-strict: include **zeros**
8. **Never** include points out of the domain (zeros of the denominator)
9. **Never** include infinity

Chapter 4 — Exponential and Logarithmic Functions

Composition

1. "Composition of f and g", "f composed with g" is $(f\circ g)(x)=f(g(x))$
2. Domain: x is in the domain of g and $g(x)$ is in the domain of f
3. In general, $f\circ g\neq g\circ f$

Inverse Relation/Function

1. Inverse relation — obtained by interchanging x and y in the equation, or reflecting the graph, or interchanging elements in pairs
2. If inverse relation is a function, it is called "inverse function" and is denoted by $f^{-1}(x)$
3. $f^{-1}(x)\neq\frac{1}{f(x)}$
4. If $f(a)\neq f(b)$ for $a\neq b$, f is **one-to-one** function
5. $(f\circ f^{-1})(x)=x\quad(f^{-1}\circ f)(x)=x$
6. The graph of f^{-1} is a reflection of the graph of f across the line $y=x$
7. **Horizontal** line test can be applied to the graph of **direct** function

Finding Inverse Function

1. Replace $f(x)$ by y
2. Interchange x and y
3. Solve for y (get $y=\ldots$)
4. If solution is **single** — function
5. Replace y with $f^{-1}(x)$

Exponential Function

1. $f(x)=a^x,\quad x\in\mathbb{R},\quad a>0,\quad a\neq 1$
2. Is one-to-one
3. Domain: $\mathbb{R}$
4. Range: $(0,+\infty)$
5. Horizontal asymptote: $y=0$
6. y-intercept: $(0,1)$
7. $e=2.718281828459045\ldots$

Logarithmic Function

1. $\log_a x$ is inverse of a^x
2. Is one-to-one
3. Domain: $(0,+\infty)$
4. Range: $\mathbb{R}$
5. Vertical asymptote: $y=0$
6. x-intercept: $(1,0)$
7. Natural logarithm $\ln x=\log_e x$
8. Common logarithm $\log x=\log_{10} x$

Properties of Logarithms

1. $\log_a 1=0\quad\log_a a=1$
2. $\log_a MN=log_aM+\log_a M$
3. $\log_a M^p=p\log_a M$
4. $\log_a\frac{M}{N}=\log_a M-\log_a N$
5. $\log_a M=\frac{\log_b M}{\log_b a}$
6. $\log_a a^x=x\quad a^{\log_a x}=x$

Applications

1. Exponential growth $P(t)=P_0e^{kt}$ $k>0$ — exponential growth rate
2. Doubling time: $P(T)=2P_0$
3. Exponential decay $P(t)=P_0e^{-kt}$ $k>0$ — decay rate
4. Half-life: $P(T)=\frac{1}{2}P_0$
5. Logistic function: $P(t)=\frac{a}{1+be^{-kt}}$